Proof of Comparison Test

The character of the two series $\sum_{1}^{\infty} u_n$ and $\sum_{1}^{\infty} v_n$ will same as the series $\sum_{m}^{\infty} u_n$ and $\sum_{m}^{\infty} v_n$ respectively. Let S_n and σ_n be the nth partial sum of the two series $\sum_{m}^{\infty} u_n$ and $\sum_{m}^{\infty} v_n$ respectively.

(i) suppose the series $\sum_{1}^{\infty} v_n$ is convergent, Then the sequence $\{\sigma_n\}$ is bounded above and consequently $\exists$ a constant A, such that $\sigma_n < A \;\forall n \in N$

Therefore $S_n = u_m + u_{m+1} + u_{m+2} + \cdots + u_{m+n-1}$

$$\leq k(v_m + v_{m+1} + \cdots + v_{m+n-1})$$

$$= k\sigma_n < kA \;\forall n \in N.$$

Hence the sequence $\{S_n\}$ is bounded.

So the series $\sum_{1}^{\infty} u_n$ convergent.

(ii) Suppose the series $\sum_{1}^{\infty} v_n$ is divergent.

Then the sequence $\{\sigma_n\}$ is unbounded above and consequently $\exists$ a constant B, such that $\sigma_n > B$, $\forall n \in N$. therefore,

$$S_n = u_m + u_{m+1} + u_{m+2} + \dots + u_{m+n-1}$$
$$\geq R(v_m + v_{m+1} + \dots + v_{m+n-1})$$
$$= R\sigma_n > RB, \ \forall n \in N.$$

Hence the sequence $\{S_n\}$ is bounded above. So the series $\sum_{1}^{\infty} u_n$ diverges.